Differential equations driven by Hölder continuous functions of order greater than 1/2

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Abstract

We derive estimates for the solutions to differential equations driven by a Hölder continuous function of order $\beta > 1/2$. As an application we deduce the existence of moments for the solutions to stochastic partial differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

1 Introduction

We are interested in the solutions of differential equations on $\mathbb{R}^m$ of the form

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{1.1}$$

where the driving force $y : [0, \infty) \to \mathbb{R}^m$ is a Hölder continuous function of order $\beta > 1/2$. If the function $f : \mathbb{R}^d \to \mathbb{R}^{md}$ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$, then there is a unique solution $x : \mathbb{R}^m \to \mathbb{R}$, which has bounded $\frac{1}{\beta}$-variation on any finite interval. These results have been proved by Lyons in [2] using the p-variation norm and the technique introduced by Young in [6]. The integral appearing in (1.1) is then a Riemann-Stieltjes integral.

In [7] Zähle has introduced a generalized Stieltjes integral using the techniques of fractional calculus. This integral is expressed in terms of fractional derivative operators and it coincides with the Riemann-Stieltjes integral $\int_0^T f dg$, when the functions f and g are Hölder continuous of orders λ and μ, respectively and $\lambda + \mu > 1$ (see Proposition 2.1 below). Using this formula for the Riemann-Stieltjes integral, Nualart and Răşcanu have obtained in [3] the existence of a unique solution for a class of general differential equations that includes (1.1). Also they have proved that the solution of (1.1) is bounded on a finite interval

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$[0,T]$ by $C_1 \exp(C_2 \|y\|_{0,T,\beta}^{\kappa})$, where $\kappa > \frac{1}{\beta}$ if f is bounded and $\kappa > \frac{1}{1-2\beta}$ is f has linear growth. Here $\|y\|_{0,T,\beta}$ denotes the β-Hölder norm of y on the time interval $[0,T]$. These estimates are based on a suitable application of Gronwall's lemma. It turns out that the estimate in the linear growth case is unsatisfactory because κ tends to infinity as β tends to $1/2$.

The main purpose of this paper is to obtain better estimates for the solution x_t in the case where f is bounded or has linear growth using a direct approach based on formula (2.8). In the case where f is bounded we estimate $\sup_{0\le t\le T} |x_t|$ by

$$C\left(1+\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right)$$

and if f has linear growth we obtain the estimate

$$C_1 \exp\left(C_2\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right).$$

In Theorem 3.1 we provide explicit dependence on f and T for the constants C, C_1 and C_2.

Another novelty of this paper is that we establish the explicit dependence of the solution x_t to (1.1) on the initial condition x_0, the driving control y and the coefficient f (Theorem 3.2). Similar results are obtained for the case $1/3 < \beta < 1/2$ in a forthcoming paper [1].

As an application we deduce the existence of moments for the solutions to stochastic partial differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We also discuss the regularity of the solution in the sense of Malliavin Calculus, improving the results of Nualart and Saussereau [4], and we apply the techniques of the Malliavin calculus to establish the existence of densities under suitable non-degeneracy conditions.

2 Fractional integrals and derivatives

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a,b)$ by the

operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{2.1}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right) \tag{2.2}$$

where $a \le t \le b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

For any $\lambda \in (0, 1)$, we denote by $C^{\lambda}(a, b)$ the space of λ-Hölder continuous functions on the interval $[a, b]$. We will make use of the notation

$$\|x\|_{a,b,\beta} = \sup_{a \le \theta < r \le b} \frac{|x_r - x_\theta|}{|r - \theta|^{\beta}},$$

and

$$\|x\|_{a,b,\infty} = \sup_{a \le r \le b} |x_r|,$$

where $x : \mathbb{R}^d \to \mathbb{R}$ is a given continuous function.

Recall from [5] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then

$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a, b).$$

- If $\alpha > \frac{1}{p}$ then

$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha - \frac{1}{p}}(a, b).$$

The following inversion formulas hold:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f, \qquad \forall f \in I_{a+}^{\alpha}(L^p) \tag{2.3}$$

$$I_{a-}^{\alpha}\left(D_{a-}^{\alpha} f\right) = f, \qquad \forall f \in I_{a-}^{\alpha}(L^p) \tag{2.4}$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \quad D_{a-}^{\alpha}\left(I_{a-}^{\alpha} f\right) = f, \quad \forall f \in L^1(a, b). \tag{2.5}$$

On the other hand, for any $f, g \in L^1(a, b)$ we have

$$\int_a^b I_{a+}^{\alpha} f(t) g(t) dt = (-1)^{\alpha} \int_a^b f(t) I_{b-}^{\alpha} g(t) dt, \tag{2.6}$$

and for $f \in I_{a+}^{\alpha}(L^p)$ and $g \in I_{a-}^{\alpha}(L^p)$ we have

$$\int_a^b D_{a+}^{\alpha} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^{\alpha} g(t) dt. \tag{2.7}$$

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [6], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [7]).

Proposition 2.1 *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1-\alpha$. Then the Riemann Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f\,(t)\, D_{b-}^{1-\alpha} g_{b-}\,(t)\, dt, \tag{2.8}$$

where $g_{b-}(t) = g(t) - g(b)$.

3 Estimates for the solutions of differential equations

Suppose that $y : [0,\infty) \to \mathbb{R}^m$ is a Hölder continuous function of order $\beta > 1/2$. Fix an initial condition $x_0 \in \mathbb{R}^d$ and consider the following differential equation

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{3.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^{md}$ is given function. Lyons has proved in [2] that Equation (3.1) has a unique solution if f is continuously differentiable and it has a derivative f' which is bounded and locally Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$.

Our aim is to obtain estimates on x_t which are better than those given by Nualart and Răşcanu in [3].

Theorem 3.1 *Let f be a continuously differentiable such that f' is bounded and locally Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$.*

(i) *Assume that f is also bounded. Then, there is a constant k, which depends only on β, such that for all T,*

$$\sup_{0 \le t \le T} |x_t| \le |x_0| + kT \|f\|_{\infty} \|f'\|_{\infty}^{\frac{1-\beta}{\beta}} \|y\|_{0,T,\beta}^{\frac{1}{\beta}}. \tag{3.2}$$

(ii) *Assume that f satisfies the linear growth condition*

$$|f(x)| \le a_0 + a_1 |x|, \tag{3.3}$$

where $a_0 \ge 0$ and $a_1 \ge 0$. Then there is a constant k depending only on β, such that for all T,

$$\sup_{0 \le t \le T} |x_t| \le 2^{kT\left[\|f'\|_{\infty} \vee a_0 \vee a_1\right]^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta}} \left(|x_0| + 1\right). \tag{3.4}$$

Proof. Without loss of generality we assume that $d = m = 1$. Assume first that f is bounded. Set $\|y\|_\beta = \|y\|_{0,T,\beta}$. Let $\alpha > 1/2$ such that $\alpha > 1-\beta$. First we use the fractional integration by parts formula given in Proposition 2.1 to obtain for all $s,t \in [0,T]$,

$$|\int_s^t f(x_r)dy_r| \le \int_s^t |D^\alpha_{s+} f(x_r)\, D^{1-\alpha}_{t-} y_{t-}(r)| dr.$$

From (2.2) and (2.1) it is easy to see

$$|D^{1-\alpha}_{t-} y_{t-}(r)| \le k\|y\|_{r,t,\beta} |t-r|^{\alpha+\beta-1} \le k\|y\|_\beta |t-r|^{\alpha+\beta-1} \tag{3.5}$$

and

$$|D^\alpha_{s+} f(x_r)| \le k \left[\|f\|_\infty (r-s)^{-\alpha} + \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}\right]. \tag{3.6}$$

Therefore

$$\begin{aligned} |\int_s^t f(x_r)dy_r| &\le k\|y\|_\beta \int_s^t \left[\|f\|_\infty (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1}\right. \\ &\quad \left. + \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\right] dr \\ &\le k\|y\|_\beta \left[\|f\|_\infty (t-s)^\beta + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^{2\beta}\right]. \end{aligned}$$

Consequently, we have

$$\|x\|_{s,t,\beta} \le k\|y\|_\beta \left[\|f\|_\infty + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^\beta\right].$$

Hence,

$$\|x\|_{s,t,\beta} \le k\|y\|_\beta \|f\|_\infty \left(1 - k\|f'\|_\infty \|y\|_\beta (t-s)^\beta\right)^{-1}.$$

Therefore,

$$\begin{aligned} \|x||_{s,t,\infty} &\le |x_s| + \|x\|_{s,t,\beta}(t-s)^\beta \\ &\le |x_s| + k\|y\|_\beta \|f\|_\infty \left(1 - k\|f'\|_\infty \|y\|_\beta (t-s)^\beta\right)^{-1} (t-s)^\beta. \end{aligned}$$

Let $A := k\|f'\|_\infty \|y\|_\beta$. Divide the interval $[0,T]$ into $n = T/\Delta$ subintervals and apply the above inequality on the interval $[0,\Delta]$, $[\Delta, 2\Delta]$ and so on recursively to obtain

$$\sup_{0\le t\le T} |x_t| \le |x_0| + kT\|f\|_\infty \|y\|_\beta (1 - A\Delta^\beta)^{-1}\Delta^{\beta-1}.$$

With the choice $\Delta = \left(\frac{1-\beta}{A}\right)^{\frac{1}{\beta}}$ we get

$$\begin{aligned} \sup_{0\le t\le T} |x_t| &\le |x_0| + kT\|f\|_\infty \|y\|_\beta \frac{1}{\beta(1-\beta)^{\frac{1-\beta}{\beta}}} \left(k\|f'\|_\infty \|y\|_\beta\right)^{\frac{1-\beta}{\beta}} \\ &= |x_0| + kT\|f'\|_\infty^{\frac{1-\beta}{\beta}} \|y\|_\beta^{\frac{1}{\beta}}. \end{aligned}$$

This proves the inequality (3.2).

Assume now that f satisfies (3.3). In this case, instead of (3.6) we have

$$|D_{s+}^{\alpha} f(x_r)| \leq k \left[(a_0 + a_1 |x_r|) (r-s)^{-\alpha} + \|f'\|_\infty \|x\|_{s,t,\beta} (r-s)^{\beta-\alpha} \right].$$

As a consequence,

$$\|x\|_{s,t,\beta} \leq k \|y\|_\beta \left[a_0 + a_1 \|x\|_{s,t,\infty} + \|f'\|_\infty \|x\|_{s,t,\beta} (t-s)^\beta \right].$$

Or

$$\|x\|_{s,t,\beta} \leq k \|y\|_\beta \left(a_0 + a_1 \|x\|_{s,t,\infty} \right) \left(1 - k \|f'\|_\infty \|y\|_\beta (t-s)^\beta \right)^{-1}.$$

Therefore,

$$\begin{aligned} |x_t| \quad &\leq \quad |x_s| + k \|y\|_\beta \left(1 - k \|f'\|_\infty \|y\|_\beta (t-s)^\beta \right)^{-1} \\ &\quad \times \left(a_0 + a_1 \|x\|_{s,t,\infty} \right) (t-s)^\beta. \end{aligned}$$

As before, divide the interval $[0,T]$ into $n = T/\Delta$ subintervals and set $\Delta = t - s$. Denote

$$\begin{aligned} A &= k \|f'\|_\infty \|y\|_\beta \\ B &= k a_0 \|y\|_\beta \\ C &= k a_1 \|y\|_\beta \\ D &= (1 - (1 - A\Delta^\beta)^{-1} C \Delta^\beta)^{-1} \\ F &= DB(1 - A\Delta^\beta)^{-1} \Delta^\beta. \end{aligned}$$

We have

$$\begin{aligned} & \|x\|_{s,t,\infty} \left[1 - k \|y\|_\beta (1 - A\Delta^\beta)^{-1} a_1 \Delta^\beta \right] \\ \leq \quad & |x_s| + k a_0 \|y\|_\beta (1 - A\Delta^\beta)^{-1} \Delta^\beta. \end{aligned}$$

This implies

$$\sup_{0 \leq r \leq t} |x_r| \leq (1 - (1 - A\Delta^\beta)^{-1} C \Delta^\beta)^{-1} \left[\sup_{0 \leq r \leq s} |x_r| + B(1 - A\Delta^\beta)^{-1} \Delta^\beta \right].$$

Or

$$\sup_{0 \leq r \leq t} |x_r| \leq D \sup_{0 \leq r \leq s} |x_r| + F.$$

Denote

$$Z_n = \sup_{0 \leq r \leq n\Delta} |x_r|,$$

where $n = \frac{T}{\Delta}$. Then

$$Z_n \leq D Z_{n-1} + F \leq \cdots \leq D^n Z_0 + \sum_{k=0}^{n-1} D^k F.$$

This yields

$$\begin{aligned}
\sup_{0\le t\le T}|x_t| &\le (1-(1-A\Delta^\beta)^{-1}C\Delta^\beta)^{-T/\Delta}|x_0| \\
&\quad + \sum_{k=0}^{n-1}(1-(1-A\Delta^\beta)^{-1}C\Delta^\beta)^{-k-1}B(1-A\Delta^\beta)^{-1}\Delta^\beta\,.
\end{aligned}$$

Then we let Δ satisfy

$$A\Delta^\beta \le 1/3\,, C\Delta^\beta \le 1/3, B\Delta^\beta \le 1/3$$

Namely, we take

$$\Delta = \left(\frac{1}{3\,(A\vee B\vee C)}\right)^{1/\beta}\,.$$

Then

$$\begin{aligned}
\sup_{0\le t\le T}|x_t| &\le 2^{T/\Delta}(|x_0|+1) \\
&\le 2^{kT\left[\|f'\|_\infty\vee a_0\vee a_1\right]^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}}(|x_0|+1)\,.
\end{aligned}$$

The proof of the theorem is now complete. ■

Suppose now that we have two differential equations of the form

$$x_t = x_0 + \int_0^t f(x_s)dy_s,$$

and

$$\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{f}(\tilde{x}_s)\tilde{y}_s\,,$$

where y and $\widetilde{y}$ are Hölder continuous functions of order $\beta > 1/2$, and f and $\widetilde{f}$ are two functions which are continuously differentiable with Hölder continuous derivatives of order $\lambda > \frac{1}{\beta}-1$. Then, we have the following estimate.

Theorem 3.2 *Suppose in addition that f is twice continuously differentiable and f'' is bounded. Then there is a constant k such that*

$$\begin{aligned}
\sup_{0\le r\le T}|x_r-\tilde{x}_r| &\le 2^{kD^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}T} \\
&\quad \times\Bigg\{|x_0-\tilde{x}_0| + \|y\|_{0,T,\beta}\left[\|f-\tilde{f}\|_\infty + \|x\|_{0,T,\beta}\|f'-\tilde{f}'\|_\infty\right] \\
&\qquad + \left[\|f\|_\infty + \|\tilde{f}\|_\infty\|x\|_{0,T,\infty}\right]\|y-\tilde{y}\|_{0,T,\beta}\Bigg\}
\end{aligned}$$

where

$$D = \|f'\|_\infty \vee \left(\|f'\|_\infty\|y\|_{0,T,\beta} + \|f''\|_\infty(\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta})T^\beta\right).$$

Proof. Fix $s,t\in[0,T]$. Set

$$x_t-\tilde{x}_t-(x_s-\tilde{x}_s)=I_1+I_2+I_3$$

where

$$\begin{aligned}
I_1 &= \int_s^t [f(x_r)-f(\tilde{x}_r)]\,dy_r\\
I_2 &= \int_s^t \left[f(\tilde{x}_r)-\tilde{f}(\tilde{x}_r)\right]dy_r\\
I_3 &= \int_s^t \tilde{f}(\tilde{x}_r)d\left[y_r-\tilde{y}_r\right].
\end{aligned}$$

The terms I_2 and I_3 can be estimated easily.

$$|I_2|\le k\|y\|_\beta\left[\|f-\tilde{f}\|_\infty(t-s)^\beta+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right]$$

and

$$|I_3|\le k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty(t-s)^\beta+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right],$$

where $\|y\|_\beta=\|y\|_{0,T,\beta}$ and $\|y-\tilde{y}\|_\beta=\|y-\tilde{y}\|_{0,T,\beta}$. The term I_1 is a little more complicated.

$$\begin{aligned}
|I_1| &\le \int_s^t |D^\alpha_{s+}\left[f(x_r)-f(\tilde{x}_r)\right]||D^{1-\alpha}_{t-}y_{t-}(r)|dr\\
&\le k\int_s^t \|y\|_{s,t,\beta}(t-r)^{\alpha+\beta-1}\left[|f(x_r)-f(\tilde{x}_r)|(r-s)^{-\alpha}\right.\\
&\quad +\|f'\|_\infty\|x-\tilde{x}\|_{s,r,\beta}(r-s)^{\beta-\alpha}\\
&\quad \left.+\|f''\|_\infty\|x-\tilde{x}\|_{s,r,\infty}\left[\|x\|_{s,r,\beta}+\|\tilde{x}\|_{s,r,\beta}\right](r-s)^{\beta-\alpha}\right]dr\\
&\le k\|y\|_\beta\left\{\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}(t-s)^\beta+\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right.\\
&\quad \left.+\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^{2\beta}\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x-\tilde{x}\|_{s,t,\beta} \le\ & k\|y\|_\beta\left\{\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}+\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right.\\
&\quad +\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta\\
&\left.+\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right\}\\
&+k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right].
\end{aligned}$$

Rearrange it to obtain

$$\|x-\tilde{x}\|_{s,t,\beta} \le k(1-k\|f'\|_\infty\|y\|_\beta(t-s)^\beta)^{-1}\left\{\|y\|_\beta\left[\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\right.\right.$$

$$
\begin{aligned}
&+\|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta \\
&\left.+\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right] \\
&\left.+k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right]\right\}.
\end{aligned}
$$

Set $\Delta = t-s$, and $A = k\|f'\|_\infty\|y\|_\beta$. Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le |x_s-\tilde{x}_s|+\|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta \\
&\le |x_s-\tilde{x}_s|+k(1-A\Delta^\beta)^{-1}\Delta^\beta\left\{\|y\|_\beta\left[\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\right.\right. \\
&\quad +\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right]\Delta^\beta \\
&\quad \left.+\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right] \\
&\quad \left.+k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right]\right\}.
\end{aligned}
$$

Denote

$$
B = k\|y\|_\beta\left(\|f'\|_\infty+\|f''\|_\infty(\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta})T^\beta\right).
$$

Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le \left(1-(1-A\Delta^\beta)^{-1}\Delta^\beta B\right)^{-1} \\
&\quad \times\left\{|x_s-\tilde{x}_s|+k(1-A\Delta^\beta)^{-1}\Delta^\beta\right. \\
&\quad \times\left[\|y\|_\beta\left[\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right]\right. \\
&\quad \left.\left.+\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right]\right]\right\}.
\end{aligned}
$$

Let Δ satisfy

$$
A\Delta^\beta \le 1/3\,,\quad B\Delta^\beta \le 1/3
$$

Namely, we take

$$
\Delta = \left(\frac{1}{3\,(A\vee B)}\right)^{1/\beta}.
$$

Then

$$
\|x-\tilde{x}\|_{s,t,\infty} \le 2\left[|x_s-\tilde{x}_s|+C\Delta^\beta\right],
$$

where

$$
\begin{aligned}
C &= \frac{3}{2}k\Bigg[\|y\|_\beta\left[\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right] \\
&\quad +\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}\Delta^\beta\right]\Bigg].
\end{aligned}
$$

Applying the above estimate recursively we obtain

$$
\sup_{0\le r\le T}|x_r-\tilde{x}_r|\le 2^n\left[|x_0-\tilde{x}_0|+C\Delta^\beta\right],
$$

where $T=n\Delta$. Or we have

$$
\begin{aligned}
\sup_{0\le r\le T}|x_r-\tilde{x}_r| &\le 2^{k(\|f'\|_\infty\vee\left(\|f'\|_\infty+\|f''\|_\infty(\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta})T^\beta\right))^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}T} \\
&\quad\times\Bigg\{|x_0-\tilde{x}_0|+\|y\|_{0,T,\beta}\left[\|f-\tilde{f}\|_\infty+\|x\|_{0,T,\beta}\|f'-\tilde{f}\|_\infty\right] \\
&\qquad+\left[\|f\|_\infty+\|\tilde{f}\|_\infty\|x\|_{0,T,\infty}\right]\|y-\tilde{y}\|_{0,T,\beta}\Bigg\}.
\end{aligned}
$$

■

4 Stochastic differential equations driven by a fBm

Let $B=\{B_t,t\ge 0\}$ be an m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H>1/2$. That is, B is a Gaussian centered process with the covariance function $E(B_t^iB_s^j)=R_H(t,s)\delta_{ij}$, where

$$
R_H(t,s)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).
$$

Consider the stochastic differential equation

$$
X_t=X_0+\int_0^t\sigma(X_s)dB_s. \tag{4.1}
$$

This equation has a unique solution (see [2] and [3]) provided σ is continuously differentiable, and σ' is bounded and Hölder continuous of order $\lambda>\frac{1}{H}-1$. The stochastic integral is interpreted as a path-wise Riemann-Stieltjes integral.

Then, using the estimate (3.4) in Theorem 3.1 we obtain the following estimate for the solution of Equation (4.1), if we choose $\beta\in\left(\frac{1}{2},H\right)$. Notice that $\frac{1}{\beta}<2$.

$$
\sup_{0\le t\le T}|X_t|\le 2^{kT\left(\|\sigma'\|_\infty\vee|\sigma(0)|\right)\|B\|_{0,T,\beta}^{1/\beta}}\left(|X_0|+1\right). \tag{4.2}
$$

If σ is bounded we can make use of the estimate (3.2) and we obtain

$$\sup_{0\leq t\leq T}|X_t| \leq |X_0| + kT\|\sigma\|_\infty\|\sigma'\|_\infty^{\frac{1-\beta}{\beta}}\|B\|_{0,T,\beta}^{\frac{1}{\beta}}. \tag{4.3}$$

These estimates improve those obtained by Nualart and Răşcanu in [3] based on a suitable version of Gronwall's lemma. The estimates (4.2) and (4.3) allow us to establish the following integrability properties for the solution of Equation (4.1).

Theorem 4.1 *Consider the stochastic differential equation (4.1). If σ' is bounded and Hölder continuous of order $\lambda > \frac{1}{H} - 1$, then*

$$E\left(\sup_{0\leq t\leq T}|X_t|^p\right) < \infty \tag{4.4}$$

for all $p \geq 2$. If furthermore σ is bounded, then

$$E\left(\exp\lambda\left(\sup_{0\leq t\leq T}|X_t|^\gamma\right)\right) < \infty \tag{4.5}$$

for any $\lambda > 0$ and $\gamma < 2\beta$.

If we apply these results to the linear equation satisfied by the derivative in the sense of Malliavin calculus of X_t then we get that X_t belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p \geq 2$. This implies that if the coefficient σ is infinitely differentiable with bounded derivatives of all orders, then, X_t belongs to $\mathbb{D}^\infty$. This allows us to deduce the regularity of the density of the random vector X_t at a fixed time $t > 0$ assuming the following nondegeneracy condition:

(H) The vector space spanned by $\left\{\left(\sigma^{ij}(X_0)\right)_{1\leq i\leq d}, 1 \leq j \leq m\right\}$ is $\mathbb{R}^m$.

That is, we have the following result.

Theorem 4.2 *Consider the stochastic differential equation (4.1). Suppose that σ is infinitely differentiable with bounded derivatives of all orders, and the assumption (H) holds. Then, for any $t > 0$ the probability law of X_t has an C^∞ density.*

In [4] Nualart and Saussereau have proved that the random variable X_t belongs locally to the space $\mathbb{D}^\infty$, and, as a consequence, they have derived the absolute continuity of the law of X_t under the assumption (H).

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